

P. S. Since the writing of the above, I have been inform'd, that, in two or three houfes, finging birds, which were at the time roosting in their cages, were thrown off their perches by the suddenness of the flock.

LXXVI. *An Investigation of some Theorems which suggest some remarkable Properties of the Circle, and are of Use in resolving Fractions, whose Denominators are certain Multinomials, into more simple ones. By Mr. John Landen.*

Read May 2, 1754. **T**HAT the principal theorems, below investigated, will be of considerable use in the doctrine of fluxions, by rendering, in many cases, the business of computing fluents more easy, will, on perusal, be obvious to every one acquainted with that branch of science. Therefore, as the facilitating computations in that doctrine (which affords us wonderful assistance in many physical enquiries) may be a means of extending our knowledge in philosophy; it is presum'd, that this paper will not be thought unworthy the notice of the Royal Society.

I.

Supposing  $\frac{x \dot{x}}{\sqrt{x^2-1}} = \frac{y \dot{y}}{\sqrt{y^2-1}}$ , where  $\dot{x}$  and  $\dot{y}$  denote the fluxions of the variable quantities  $x$  and  $y$

re-

respectively, and  $n$  an invariable quantity; it is propos'd to find, in terms of  $y$  and  $x$ , the equation of which  $x$  is a root, and  $x^2 - 2xz + 1 = 0$ , a divisor.

Taking the fluents of the given fluxionary equation, we have, supposing  $x = 1$  when  $y = 1$ , hyp.

log. of  $\sqrt[n]{x^2 - 1} = \text{hyp. log. of } y + \sqrt{y^2 - 1}$ , or

$\sqrt[n]{x^2 - 1} = y + \sqrt{y^2 - 1}$ : Whence, by substituting for  $x$  its value  $\frac{z^2 + 1}{2z}$  (found by the equation

$x^2 - 2xz + 1 = 0$ ), we have  $z^n = y + \sqrt{y^2 - 1}$ :

Therefore  $z^n - y$  is  $= \sqrt{y^2 - 1}$ ; and, squaring both sides,  $z^{2n} - 2yz^n + y^2 = y^2 - 1$ . Consequently

$z^{2n} - 2yz^n + 1 = 0$ ; which, supposing  $n$  a positive integer, is the equation sought.

Now it is obvious,  $n$  being such an integer, that this equation will have as many trinomial divisors, of the form  $x^2 - 2xz + 1$ , as there are values of  $x$  corresponding to a given value of  $y$ : Which values of  $x$ , when  $y$  is not greater than 1, nor less than  $-1$  (the only case I purpose to consider), will not be readily

obtain'd from the equation  $\sqrt[n]{x^2 - 1} = y + \sqrt{y^2 - 1}$  found above: But, if we multiply the given fluxionary equation by  $\frac{1}{\sqrt{-1}}$ , we get  $\frac{nz}{\sqrt{1 - x^2}} = \frac{y}{\sqrt{1 - y^2}}$ ;

of which the equation of the fluents is  $n \times \text{circ. arc rad. } 1. \text{ cosine } x = \text{circ. arc rad. } 1. \text{ cosine } y$ ; where  $x$  is  $= 1$  when  $y = 1$ , agreeable to the supposition we

made

made



III.

If  $y$  be  $= -1$ ; then,  $A$  being  $= \frac{C}{2}$ ;  $p, q, r, s, \&c.$  will be the cosines of  $\frac{C}{2n}, \frac{3C}{2n}, \frac{5C}{2n}, \&c. (n)$  respectively: Therefore, if  $n$  be an odd number, one of those arcs will be  $\frac{C}{2}$ , whose cosine is  $-1$ .

IV.

If in the equations  $z^{2n} - 2y z^n + 1 = 0$ , and  $x^2 - 2x z + 1 = 0$ , we substitute  $v-1$  for  $z$ , they

become  $\overline{v-1}^{2n} - 2y \times \overline{v-1}^n + 1 = 0$ , and  $\overline{v-1}^2 - 2x \times \overline{v-1} + 1 = v^2 - 2 + 2x \times v + 2 + 2x = 0$ .  
Consequently

$$\left. \begin{aligned} v^{2n} - 2n v^{2n-1} + \dots + 2n \times \frac{2n-1}{2} v^2 - 2nv + 1 \\ \dots + 2yn \times \frac{n-1}{2} v^2 + 2ynv + 2y \\ + 1 \end{aligned} \right\} =$$

$\overline{v^2 - 2 + 2p \times v + 2 + 2p \times v^2 - 2 + 2q \times v + 2 + 2q \times v^2 - 2 + 2r \times v + 2 + 2r \times \&c. (n)}$ ; where, of the two signs prefix'd to the terms where  $y$  is a factor, the upper or lower takes place, according as  $n$  is an even or an odd number. Whence, by the nature of equations, it follows, that  $2 + 2p \times 2 + 2q \times 2 + 2r, \&c.$  is  $= 2 + 2y$ . But this equation vanishing when  $y$  is  $= 1$  and  $n$  an even number, or when  $y$  is  $= -1$

and  $n$  an odd number, it will be proper to consider those two cases more particularly.

V.

First, Let us suppose  $y = 1$ , and  $n$  an even number: Then  $p$  being  $= 1$ , and one of the other cosines  $q, r, s, \&c. = -1$  (*Art. II.*); we shall have

$$\frac{v^{2^n} - 2n v^{2^{n-1}} + \dots + n^2 v^2 = \overline{v^2 + 0 \times v^2 - 4v + 4 \times v^2 - 2 + 2q \times v + 2 + 2q \times v^2 - 2 + 2r \times v + 2 + 2r, \&c.}}{v^2}$$

Therefore dividing by  $v^2$ ,

$$\frac{v^{2^{n-2}} - 2n v^{2^{n-3}} + \dots + n^2 = \overline{v^2 - 4v + 4 \times v^2 - 2 + 2q \times v + 2 + 2q \times v^2 - 2 + 2r \times v + 2 + 2r, \&c.}}{v^2}$$

that factor in which the value of the cosine  $q$ , or  $r$ , &c. is  $-1$ , being expung'd.

Consequently  $n^2$  is  $= 4 \times 2 + 2q \times 2 + 2r \times 2 + 2s, \&c.$  when the factor, whose value is nothing, is expung'd.

VI.

Let us now suppose  $y = -1$ , and  $n$  an odd number: Then one of the cosines  $p, q, r, \&c.$  being  $= -1$  (*Art. III.*),

$$\frac{v^{2^n} - 2n v^{2^{n-1}} + \dots + n^2 v^2 \text{ will be } = \overline{v^2 + 0 \times v^2 - 2 + 2p \times v + 2 + 2p \times v^2 - 2 + 2q \times v + 2 + 2q, \&c.}}{v^2}$$

Therefore, dividing by  $v^2$ ,

$$\frac{v^{2^{n-2}} - 2n v^{2^{n-3}} + \dots + n^2 \text{ will be } = \overline{v^2 - 2 + 2p \times v + 2 + 2p \times v^2 - 2 + 2q \times v + 2 + 2q, \&c.}}{v^2}$$

and consequently  $n^2 = 2 + 2p \times 2 + 2q \times 2 + 2r, \&c.$  when the factor, whose value is nothing, is expung'd.

VII.

VII.

Substituting in the equations  $z^{2n} - 2yz^n + 1 = 0$ ,  
and  $z^2 - 2xz + 1 = 0$ ,  $\frac{a+\omega}{a-\omega}$  instead of  $z$ , we have

$$\left(\frac{a+\omega}{a-\omega}\right)^{2n} - 2y \times \left(\frac{a+\omega}{a-\omega}\right)^n + 1$$

$$= \frac{\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega}^n \times \overline{a-\omega}^n + \overline{a-\omega}^{2n}}{\overline{a-\omega}^{2n}} = 0, \text{ and}$$

$$\left(\frac{a+\omega}{a-\omega}\right)^2 - 2x \times \frac{a+\omega}{a-\omega} + 1$$

$$= \frac{\overline{a+\omega}^2 - 2x \times \overline{a+\omega} \times \overline{a-\omega} + \overline{a-\omega}^2}{\overline{a-\omega}^2}$$

$$= \frac{2 + 2x \times \omega^2 + \frac{1-x}{1+x} a^2}{\overline{a-\omega}^2} = 0. \text{ Consequently}$$

$$\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega}^n \times \overline{a-\omega}^n + \overline{a-\omega}^{2n} \text{ will be } =$$

$$\overline{2 + 2p \times 2 + 2q \times 2 + 2r, \&c.} \times \omega^2 + \frac{1-p}{1+p} a^2$$

$$\times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2, \&c.$$

But, by *Art.* IV.  $\overline{2 + 2p \times 2 + 2q \times 2 + 2r, \&c.}$   
is  $= 2 \overline{+} 2y$ , the upper or lower of the two figs  
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prefix'd to  $y$  taking place according as  $n$  is an even or an odd number.

Therefore  $\frac{a+\omega}{\omega^2 + 2y \times \frac{a+\omega}{\omega^2} \times a - \omega + a - \omega}^{2n}$  is =  
 $\frac{2 + 2y \times \omega^2 + \frac{1-p}{1+p} a^2 \times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2}{\omega^2 + 2y \times \omega^2 + \frac{1-p}{1+p} a^2 \times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2}$ ,  
 &c.

Now  $p$  being the cosine of any number of degrees, radius being 1,  $\frac{1-p}{1+p} a^2$  will be the square of the tangent of half so many degrees, radius being  $a$ : Therefore, denoting that tangent by  $b$ ; and the tangents of half the arcs describ'd with the radius  $a$ , whose cosines, when the radius is 1, are  $q, r, s$ , &c. being denoted by  $c, d, e$ , &c. respectively; we have

$\frac{a+\omega}{\omega^2 + 2y \times \frac{a+\omega}{\omega^2} \times a - \omega + a - \omega}^{2n} = \frac{2 \mp 2y \times \omega^2 + b^2 \times \omega^2 + c^2 \times \omega^2 + d^2}{\omega^2 + 2y \times \omega^2 + \frac{1-p}{1+p} a^2 \times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2}$ , &c. But when  $y$  is = 1, and  $n$  an even number; or  $y = -1$ , and  $n$  an odd number;  $2 \mp 2y$  being = 0; nothing can be determin'd from that equation: Wherefore, in those cases, recourse must be had to what is done above.

VIII.

Let us suppose  $y = 1$ , and  $n$  an even number :

Then the equation  $\frac{a+\omega}{\omega^2 + 2y \times \frac{a+\omega}{\omega^2} \times a - \omega + a - \omega}^{2n}$   
 $+ \frac{a - \omega}{\omega^2 + 2y \times \frac{a - \omega}{\omega^2} \times a + \omega + a + \omega}^{2n} = \frac{2 + 2p \times 2 + 2q \times 2 + 2r}{\omega^2 + 2y \times \omega^2 + \frac{1-p}{1+p} a^2 \times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2}$ , &c.

becomes

becomes  $\frac{a+\omega}{a+\omega}^{2n} - 2 \times \frac{a+\omega}{a+\omega}^n \times \frac{a-\omega}{a-\omega}^n + \frac{a-\omega}{a-\omega}^{2n} = 4$   
 $\times \frac{2+2q}{2+2q} \times \frac{2+2r}{2+2r}, \&c. \omega^2 \times \omega^2 + \frac{1-q}{1+q} a^2 \times \omega^2 + \frac{1-r}{1+r} a^2,$

&c.  $p$  being  $= 1$  (*Art.* II.) and  $\frac{1-p}{1+p} a^2 (= b^2) = 0.$

Moreover, one of the other cosines  $q, r, s, \&c.$  being  $= -1$  (*Art.* II.), some one of the factors  $2 + 2q, 2 + 2r, 2 + 2s, \&c.$  will vanish: Which factor being expung'd from the product  $4 \times \frac{2+2q}{2+2q} \times \frac{2+2r}{2+2r},$  &c. and restor'd to the divisor  $\omega^2 + \frac{1-q}{1+q} a^2,$  or

$\omega^2 + \frac{1-r}{1+r} a^2, \&c.$  from which it was taken, that divisor will become  $4 a^2;$  and the product  $4 \times \frac{2+2q}{2+2q} \times \frac{2+2r}{2+2r}, \&c.$  will then (by *Art.* V.) be  $= n^2.$

Consequently  $\frac{a+\omega}{a+\omega}^{2n} - 2 \times \frac{a+\omega}{a+\omega}^n \times \frac{a-\omega}{a-\omega}^n + \frac{a-\omega}{a-\omega}^{2n},$  will be  $= n^2 \times \omega^2 \times 4 a^2 \times \omega^2 + c^2 \times \omega^2 + d^2, \&c.$  where the factor  $4 a^2$  takes place instead of  $\omega^2 + sq.$  of the tang. of  $90^\circ.$

If  $y$  be  $= 1,$  and  $n$  an odd number,  $p$  will be  $= 1,$  and  $b = 0;$  but no one of the cosines  $q, r, s, \&c.$  will be  $= -1,$  as when  $n$  is an even number. There-

fore, in this case, the equation  $\frac{a+\omega}{a+\omega}^{2n} - 2 y \times \frac{a+\omega}{a+\omega}^n$   
 $\times \frac{a-\omega}{a-\omega}^n + \frac{a-\omega}{a-\omega}^{2n} = \frac{2+2y}{2+2y} \times \frac{\omega^2+b^2}{\omega^2+b^2} \times \frac{\omega^2+c^2}{\omega^2+c^2}, \&c.$   
 becomes  $\frac{a+\omega}{a+\omega}^{2n} - 2 \times \frac{a+\omega}{a+\omega}^n \times \frac{a-\omega}{a-\omega}^n + \frac{a-\omega}{a-\omega}^{2n} =$   
 $4 \times \omega^2 \times \frac{\omega^2+c^2}{\omega^2+c^2} \times \frac{\omega^2+d^2}{\omega^2+d^2}, \&c.$



IX.

By taking the square root of  $\overline{a + \omega}^{2n} - 2 \times \overline{a + \omega}^n \times \overline{a - \omega}^n + \overline{a - \omega}^{2n}$ , and of its two values just now found, we have, when  $n$  is an even number,  $\overline{a + \omega}^n - \overline{a - \omega}^n = 2 a n \omega \times \sqrt{\omega^2 + c^2} \times \sqrt{\omega^2 + d^2}$ , &c.  $2 a$  taking place instead of  $\sqrt{\omega^2 + \text{sq. of the tang. of } 90^\circ}$ .

And, when  $n$  is an odd number,  $\overline{a + \omega}^n - \overline{a - \omega}^n = 2 \omega \times \sqrt{\omega^2 + c^2} \times \sqrt{\omega^2 + d^2}$ , &c. Whence the following construction is inferr'd.

X.

Describe about the centre  $C$  (*Plate XX. fig. 1. and 2.*), with the radius  $a$ , the circle  $P A' A'' A'''$ , &c. ; draw the diameter  $PC \mathcal{Q}$ , and the tangent  $B''' P B'$ ; divide the semicircumference  $P A' \mathcal{Q}$  into as many equal parts  $P A', A' A'', A'' A'''$ , &c. as there are units in the integer  $n$ ; draw the secants  $C A' B', C A'' B''$ , &c. and, taking on  $C \mathcal{Q}$  any point  $O$ , draw  $K''' O K'$  parallel to  $B''' P B'$ ; likewise draw  $B' K', B'' K'', B''' K'''$ , &c. parallel to  $P \mathcal{Q}$ ; and call  $CO, \omega$ .

Then will  $q$  be the cosine of twice the angle  $PC A$ ,  $r$  the cosine of twice  $PC A''$ ,  $s$  the cosine of twice  $PC A'''$ , &c. if the radius be 1.

Therefore  $P B' = O K'$  will be  $= c$ ,  $P B'' = O K'' = d$ , &c. and  $C K' = \sqrt{\omega^2 + c^2}$ ,  $C K'' = \sqrt{\omega^2 + d^2}$ ,

&c. Consequently  $OP^n - O \mathcal{Q}^n$  being  $= \overline{a + \omega}^n$

$\overline{a-\omega}^n$ , and  $n \times P \mathcal{Q} \times C O \times C K' \times C K''$ , &c. =  $2 a n \omega \times \sqrt{\omega^2 + c^2} \times \sqrt{\omega^2 + d^2}$ , &c. when  $n$  is an even number;  $O P^n - O \mathcal{Q}^n$  will then be =  $n \times P \mathcal{Q} \times C O \times C K' \times C K''$ , &c. where the diameter  $P \mathcal{Q}$

takes place instead of the infinite quantity  $C K^{\frac{n}{2}}$ .

But if  $n$  be an odd number,  $O P^n - O \mathcal{Q}^n$  will be =  $2 \times C O \times C K' \times C K'' \times C K'''$ , &c.

XI.

It is evident that, of the factors  $C K'$ ,  $C K''$ ,  $C K'''$ , &c. the first and last, the second and last but one, &c. are respectively equal to each other. Therefore, omitting the squares of the factors below  $P \mathcal{Q}$ , and the squares of their values,

$O P^n - O \mathcal{Q}^n$  is =  $n \times P \mathcal{Q} \times C O \times C K'^2 \times C K''^2 \times$

$C K'''^2$ , &c. and  $\overline{a+\omega}^n - \overline{a-\omega}^n = 2 a n \omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$ , &c. when  $n$  is an even number; or

$O P^n - O \mathcal{Q}^n$  is =  $2 \times C O \times C K'^2 \times C K''^2 \times C K'''^2$ ,

&c. and  $\overline{a+\omega}^n - \overline{a-\omega}^n = 2 \omega \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$ , &c. when  $n$  is an odd number.

XII.

If we suppose  $y = -1$ , and  $n$  an odd number, it will appear, by proceeding much in the same manner

as in *Art. VIII.* that  $\overline{a+\omega}^{2n} + 2 \times \overline{a+\omega}^n \times \overline{a-\omega}^n +$

$\overline{a-\omega}^{2n}$  is =  $n^2 \times 4a^2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2} \times \overline{\omega^2 + d^2}$ , &c.

&c. where the factor  $4 a^2$  takes place instead of  $\omega^2 + \text{sq. of the tang. of } 90^\circ$ .

If  $y$  be  $= -1$ , and  $n$  an even number,  $\overline{a + \omega}^n + 2 \times \overline{a + \omega}^n \times \overline{a - \omega}^n + \overline{a - \omega}^{2n}$  is  $= 4 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$ , &c.

Whence, by extracting the square root of both sides of those equations, we have, when  $n$  is an odd number,  $\overline{a + \omega}^n + \overline{a - \omega}^n = 2 a n \times \sqrt{\overline{\omega^2 + b^2}} \times \sqrt{\overline{\omega^2 + c^2}}$ , &c.  $2 a$  taking place instead of  $\sqrt{\overline{\omega^2 + \text{sq. of the tang. of } 90^\circ}}$ : And, when  $n$  is an even number,  $\overline{a + \omega}^n + \overline{a - \omega}^n = 2 \times \sqrt{\overline{\omega^2 + b^2}} \times \sqrt{\overline{\omega^2 + c^2}}$ , &c. Hence we infer this construction.

### XIII.

Having describ'd about the centre  $C$  (*fig. 3. and 4.*), with the radius  $a$ , the circle  $P a' A' a'' A''$ , &c. draw the diameter  $PC Q$ , and the tangent  $b'' P b^+$ ; divide the semicircumference  $P a' Q$  into as many equal parts  $P a', a' A', A' a''$ , &c. as there are units in  $2 n$ ; draw the secants  $C a' b', C a'' b''$ , &c. and, thro' any point ( $O$ ) in  $C Q$ , draw  $k'' O k^+$  parallel to  $b'' P b^+$ ; likewise draw  $b' k', b'' k''$ , &c. parallel to  $P Q$ ; and call  $CO$ ,  $\omega$ .

Then, if the radius be  $1$ ,  $p$  will be the cosine of twice the angle  $PC a'$ ,  $q$  the cosine of twice  $PC a''$ , &c. therefore  $P b' = O k'$  will be  $= b$ ,  $P b'' = O k'' = c$ , &c. and  $C k' = \sqrt{\overline{\omega^2 + b^2}}$ ,  $C k'' = \sqrt{\overline{\omega^2 + c^2}}$ , &c.

Con-

Consequently  $O P^n + O Q^n$  being  $= \overline{a+\omega}^n + \overline{a-\omega}^n$ ,  
 and  $n \times P Q \times C k' \times C k''$ , &c.  $= 2an \times \sqrt{\omega^2 + b^2}$   
 $\times \sqrt{\omega^2 + c^2}$ , &c. when  $n$  is an odd number;  $O P^n +$   
 $O Q^n$  will then be  $= n \times P Q \times C k' \times C k''$ , &c. where  
 the diameter  $P Q$  takes place instead of the infinite  
 quantity  $C k^{\frac{n+1}{2}}$ .

But if  $n$  be an even number,  $O P^n + O Q^n$  will be  
 $= 2 \times C k' \times C k''$ , &c.

XIV.

It is obvious that, of the factors  $C k'$ ,  $C k''$ , &c. the  
 first and last, the second and last but one, &c. are re-  
 spectively equal to each other: Therefore the squares  
 of the factors below  $P Q$ , and the squares of their  
 values, being omitted,

$O P^n + O Q^n$  is  $= n \times P Q \times C k'^2 \times C k''^2$ , &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2an \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$ , &c.  
 when  $n$  is an odd number; or

$O P^n + O Q^n$  is  $= 2 \times C k'^2 \times C k''^2$ , &c. and

$\overline{a+\omega}^n + \overline{a-\omega}^n = 2 \times \overline{\omega^2 + b^2} \times \overline{\omega^2 + c^2}$ , &c. when  
 $n$  is an even number.

XV.

Writing in the equation  $\overline{a+\omega}^{2n} - 2y \times \overline{a+\omega} \times$   
 $\overline{a-\omega}^{2n} = \overline{2 + 2y \times \omega^2 + b^2} \times \overline{\omega^2 + c^2}$ , &c.  
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(found by *Art.* VII.)  $a - u$  for  $\omega$ , the same becomes

$$\frac{2a - u - 2yu^n}{u^2 - 2au + a^2 + b^2} \times \frac{2a - u + u^{2n}}{u^2 - 2au + a^2 + c^2}, \&c.$$

$$= \frac{2 + 2y \times u^2 - 2au + \beta^2 \times u^2 - 2au + \gamma^2 \times u^2 - 2au + \delta^2}{u^2 - 2au + \delta^2}, \&c.$$

if instead of  $\sqrt{a^2 + b^2}$ ,  $\sqrt{a^2 + c^2}$ , &c. (the secants of the arcs of which  $b, c, d, \&c.$  are tangents), we put  $\beta, \gamma, \delta, \&c.$

And, by a like substitution in the equations in *Art.* XI. and XIV. it appears, that

$$\frac{2a - u - u^n}{u^2 - 2au + \delta^2}, \&c. \text{ or } 2 \times \frac{a - u \times u^2 - 2au + \gamma^2}{u^2 - 2au + \delta^2}, \&c.$$

according as  $n$  is an even or

an odd number: And that  $\frac{2a - u + u^n}{u^2 - 2au + \beta^2} \times \frac{n^2 - 2au + \gamma^2}{u^2 - 2au + \gamma^2}, \&c.$  or  $2 \times \frac{u^3 - 2au + \beta^2 \times u^2 - 2au + \gamma^2}{u^2 - 2au + \gamma^2}, \&c.$  according as  $n$  is an odd or an even number.

From what is done above, I might now deduce many corollaries; and, by means of other substitutions, investigate other theorems; but want of leisure obliges me to desist.

